Interval Finite Element Analysis of Thin Plates

M. V. Rama Rao
Vasavi College of Engineering,
Hyderabad - 500 031 INDIA

Rafi Muhanna
School of Civil and Environmental Engineering
Georgia Institute of Technology, Atlanta, GA 30332-0355, USA

Robert L. Mullen
Department of Civil and Environmental Engineering
University of South Carolina, Columbia, SC 29208 USA

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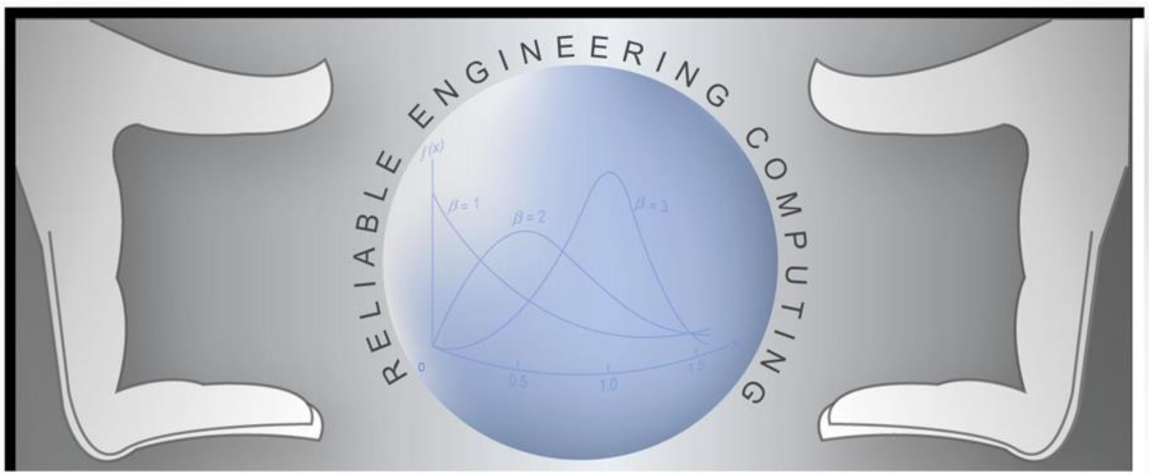

RELIABLE ENGINEERING COMPUTING
f(x)
β = 1
β = 2
β = 3
0
0.5
1.0
1.5

Outline

- Analysis of thin plates
- Present work
- Interval Finite Element Model of thin plate
- Example Problems
- Conclusions

Motivation for the present work

- To the authors' knowledge, applications of interval methods for the analysis of plates with uncertainty of load and material properties do not exist anywhere in literature.
- In view of this, we present an initial investigation into the application of interval finite element methods to problems of bending of thin plates.

Present work

- This work presents the application of interval finite element methods to the analysis of thin plates
- Uncertainty is considered in both the applied load and Young's modulus
- In the present study a clamped rectangular plate is analysed and the deformations are obtained.
- Example problems are presented and discussed

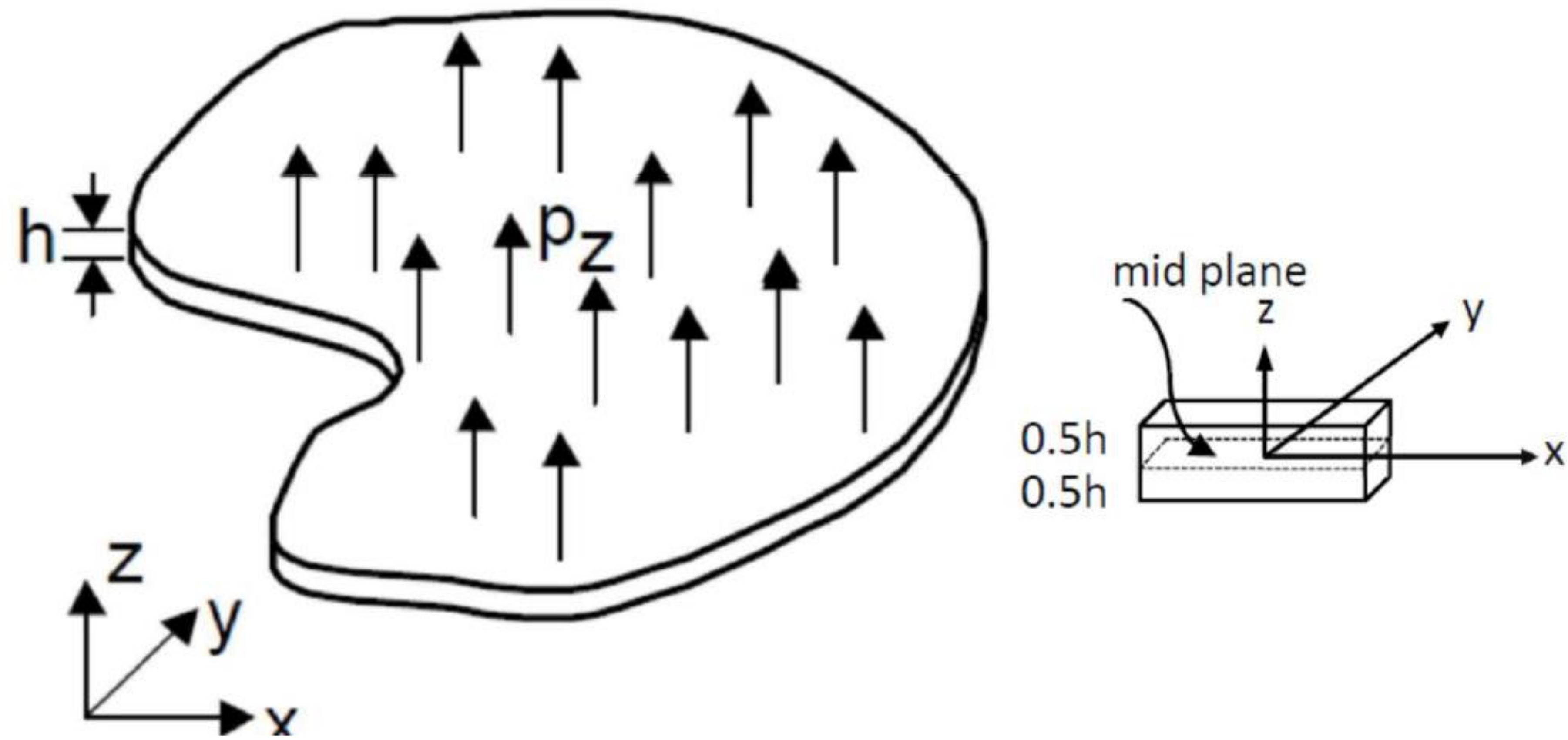
Present work

- The plate is assumed to be orthotropic. Interval uncertainty is associated with the Young's modulus of the plate and also with the applied load.
- Interval Finite Element Method (IFEM) developed in the earlier work for line elements of the authors for truss and frame structures (Rama Rao, Muhanna and Mullen, 2011)is applied to the case of thin plates in the present work.

Present work

- This method is capable of obtaining bounds for interval forces and moments with the same level of sharpness as displacements and rotations.
- Example problems of the thin plate are solved to demonstrate that the present method is capable of obtaining sharp bounds.
- Results are compared to the values of displacements and forces obtained using combinatorial and Monte Carlo solutions.

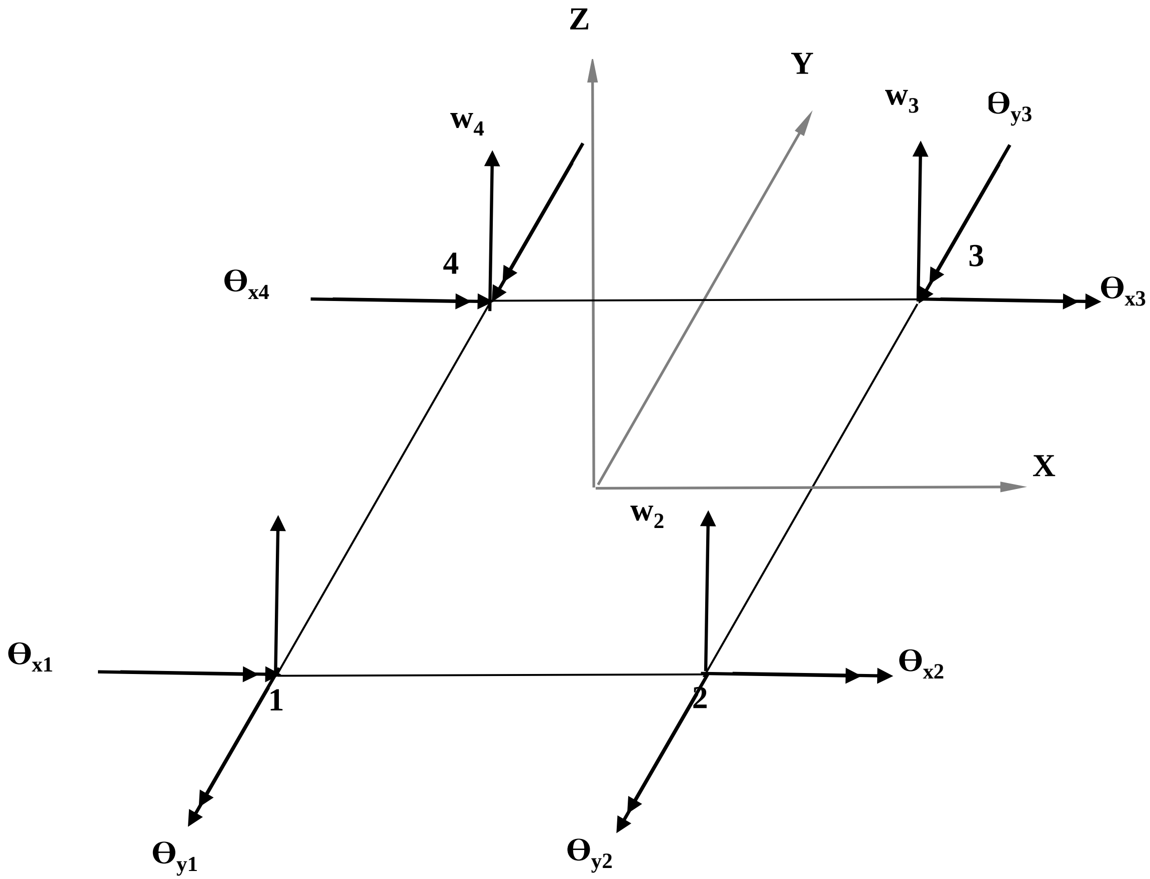
Geometry of thin plate



Geometry of thin plate

- The plate is discretized into rectangular ACM (Adini-Clough-Melosh) plate elements.
- The ACM element is a non-conforming element with 12 degrees of freedom (3 degrees of freedom at each of the four nodes)
- Degrees of freedom at each node are transverse displacement and normal rotation about each axis wz,θx and θy

Geometry of thin plate



Degrees of freedom of ACM Element

Interval FEA of plate

The stiffness matrix of the plate is expressed as

$$\left[\boldsymbol{K}^{(e)}\right] = \int_{-a}^{a}\int_{-b}^{b} \left[B\Phi^{-1}\right]^T \left[\boldsymbol{D}\right]\left[B\Phi^{-1}\right] dxdy$$

The load vector of the plate is expressed as

$$\left\{\boldsymbol{P}^{(e)}\right\} = \left[\Phi^{-1}\right]^T \left(\int_{-a}^{a}\int_{-b}^{b} \boldsymbol{p}_z \left[N(x,y)\right]^T dxdy \right)$$

Interval FEA of plate

The D matrix of the plate is expressed as

$$[\boldsymbol{D}] = \frac{Eh^3}{12(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{(1-\nu)}{2} \end{bmatrix}$$

Interval FEA of plate

D matrix is decomposed as (Xiao,Fedele and Muhanna, 2013)

$$[\boldsymbol{D}] = A_k diag(\boldsymbol{\Lambda}_k \boldsymbol{\alpha}_k) A_k^T$$

$$\text{where } \boldsymbol{\alpha}_k = \boldsymbol{E} \; ; \boldsymbol{\Lambda}_k = \left\{ \frac{h^3}{12(1-\nu^2)} \quad \frac{h^3}{12} \quad \frac{h^3}{24(1+\nu)} \right\}^T \; ; A_k = \begin{bmatrix} 1 & 0 & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Interval FEA of plate

The element stiffness matrix is decomposed as

$$\left[\boldsymbol{K}^{(e)}\right]=\left[A^{(e)}\right]diag(\Lambda\boldsymbol{\alpha})\left[A^{(e)}\right]^{T}$$

The stiffness matrix for the structure is expressed as

$$[\boldsymbol{K}]=[A][\boldsymbol{D}][A]^{T}$$

The force vector for the structure is expressed as

$$\{\boldsymbol{P}\}_{n\times 1}=\begin{Bmatrix}\boldsymbol{P}_1^{(e)}\\ \boldsymbol{P}_2^{(e)}\\ \boldsymbol{P}_3^{(e)}\\ \ldots\end{Bmatrix}=[M]_{n\times m}[\delta]_{m\times 1}$$

Interval FEA of plate

Modified potential energy Π^* can be expressed as

$$\Pi^* = \frac{1}{2}\{\boldsymbol{U}\}^T[\boldsymbol{K}]\{\boldsymbol{U}\} - \{\boldsymbol{U}\}^T\{\boldsymbol{P}\} + \lambda_1^T([C]\{\boldsymbol{U}\}-\{V\}) + \lambda_2^T([B_1]\{\boldsymbol{U}\}-\{\boldsymbol{\kappa}\})$$

where

U is the displacement vector

P is the load vector

$\boldsymbol{K}$ is the stiffness matrix

C is the constraint matrix

B1 is the strain-curvature matrix

$\boldsymbol{\kappa}$ is the vector of curvatures

Interval FEA of plate

Invoking the stationarity of Π*, we obtain

$$\left(\begin{pmatrix} \boldsymbol{0} & C^T & B_1^T & 0 \\ C & 0 & 0 & 0 \\ B_1 & 0 & 0 & -I \\ 0 & 0 & -I & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{K} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right) \begin{pmatrix} \boldsymbol{U} \\ \lambda_1 \\ \lambda_2 \\ \kappa \end{pmatrix} = \begin{pmatrix} \boldsymbol{P}_C \\ 0 \\ 0 \\ 0 \end{pmatrix} + \left\{ \begin{bmatrix} [M] \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\} \{\delta\}$$

The above equation can be solved by Neumaier's approach to obtain the interval displacements ***{U}*** and curvatures ***{κ}***

Interval FEA of plate

Vector of interval moments *{M}* is obtained from the vector of curvatures $\{\kappa\}$ as follows:

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = [D] \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} = \frac{Eh^3}{12(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{(1-\nu)}{2} \end{bmatrix} \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix}$$

Example problem

Table 1 Properties of rectangular plate and discretization scheme	
Length Lx	2.0 m
Width Ly	3.0 m
Thickness	0.025 m
Young's modulus	210 GPa
Poisson's ratio ν	0.3
Applied Pressure p_z	14.0×10^3 Pa
Number of divisions along x-axis	nx
Number of divisions along y-axis	ny
Notation for discretization scheme	$nx \times ny$

Example problem

- First the present interval approach is validated by solving the problem of a rectangular plate with a 4×4 discretization scheme.
- Solution is computed using the present interval approach and combinatorial solution.
- The computation of results for combinatorial solution required computation of results for 2^{16}=65,536 combinations

Example problem

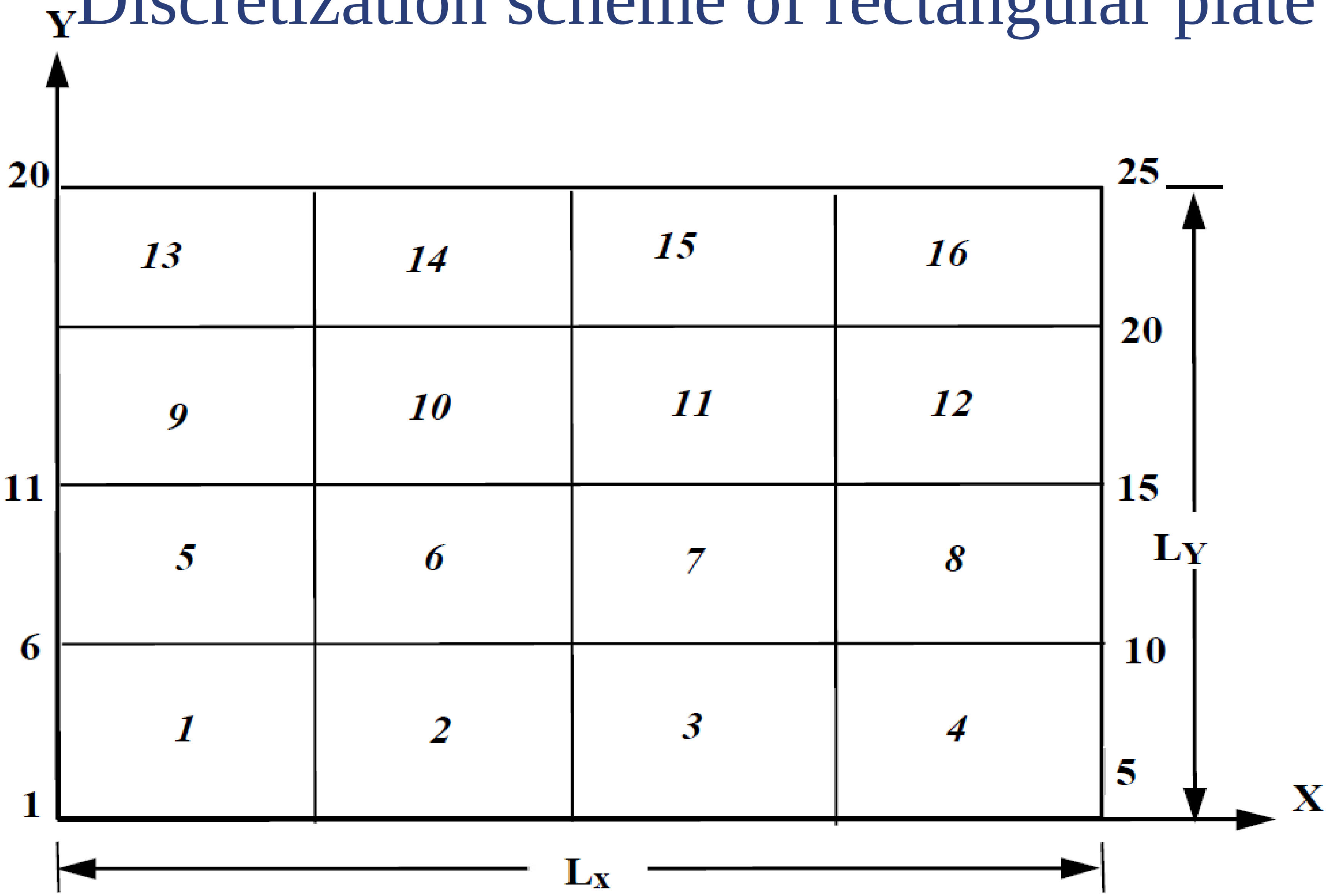
Solution is computed for the following load cases

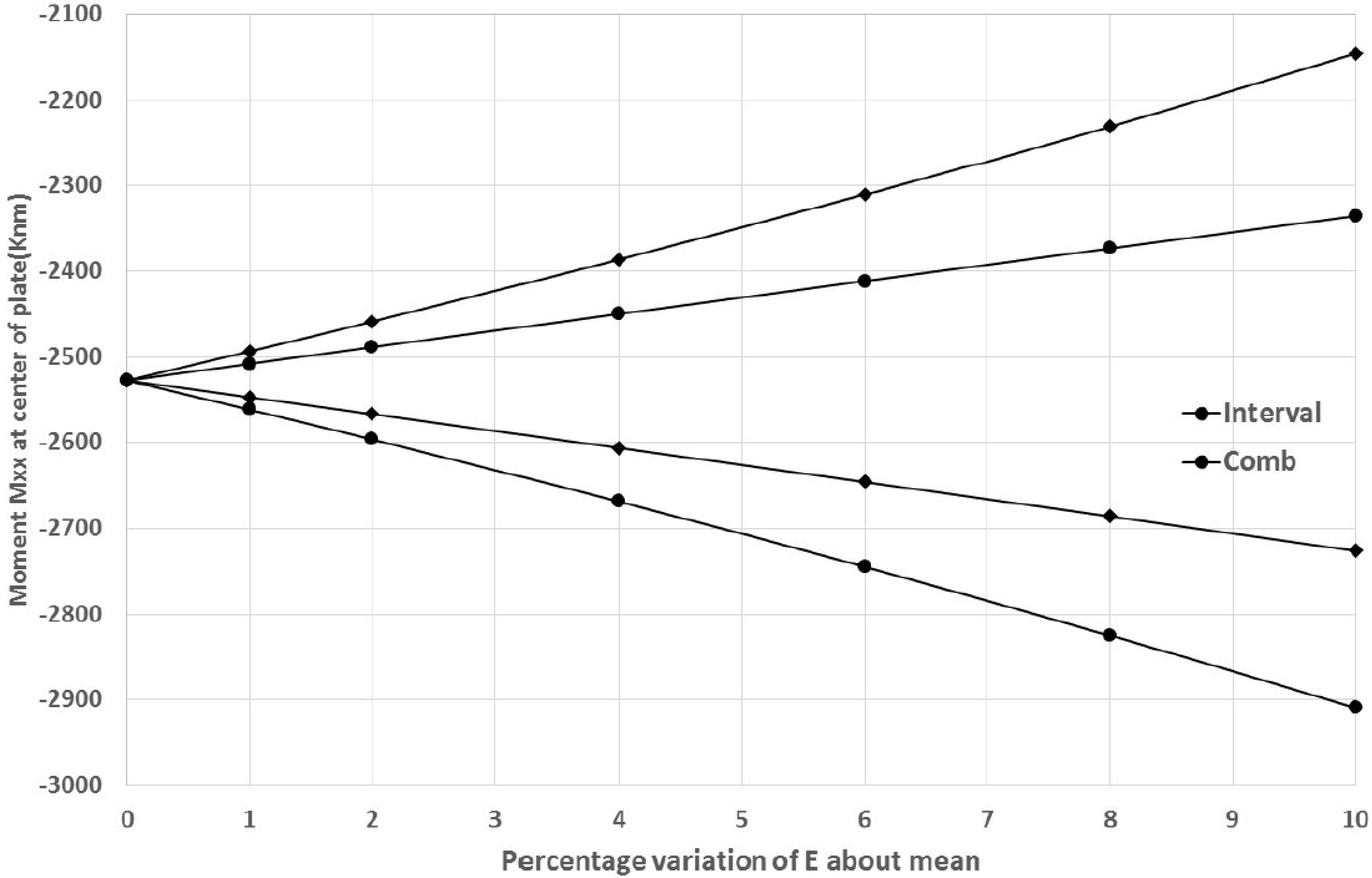
- Case A: Uncertainty of load alone
- Case B: Uncertainty of Young's modulus (E) alone
- Case C: Uncertainty of load and E

Maximum uncertainty in load is **10 percent (±5 percent about the mean value)**

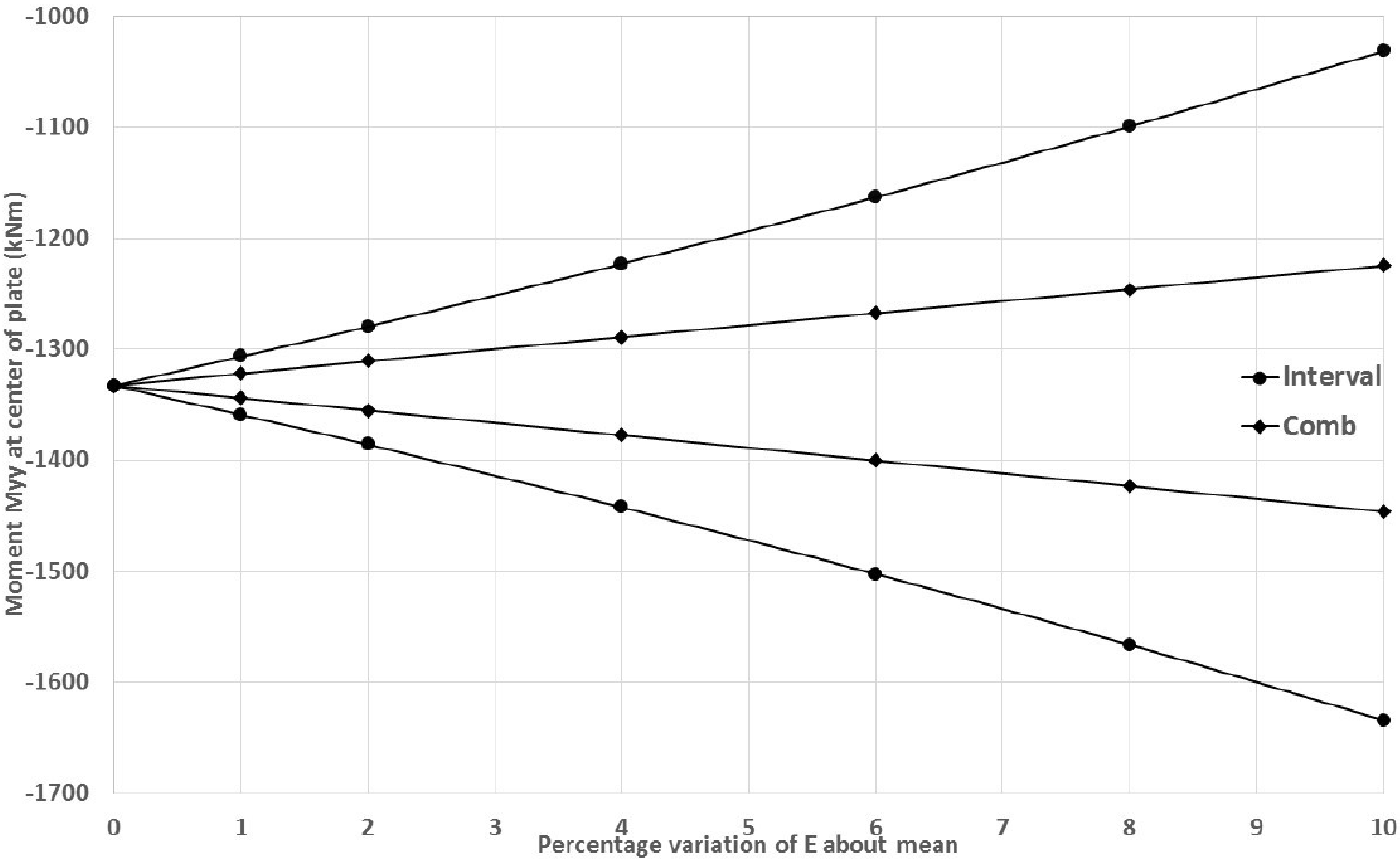
Maximum uncertainty of E is **1 percent (±0.5 percent about the mean value)**

Discretization scheme of rectangular plate





Variation of Mxx at center of plate (at node 13) w.r.t. uncertainty of E



Variation of Myy at center of plate (at node 13) w.r.t. uncertainty of E

Example problem

- It is observed from these figures that the interval values of Mxx and Myy computed enclose the combinatorial solution at all levels of uncertainty

Table 2 Clamped rectangular plate (4×4)– selected displacements and rotations of the plate for 10% uncertainty of load (Case-A)						
Method	$w_{13}\times10^3$(m)		$\theta_x\times10^3$(radians) at node 7		$\theta_y\times10^3$(radians) at node 12	
	Lower	Upper	Lower	Upper	Lower	Upper
Combinatorial	-1.90416	-1.72281	-1.10534	-0.96432	2.55516	2.82412
Interval	-1.90416	-1.72281	-1.10534	-0.96432	2.55516	2.82412
Error%	0.0	0.0	0.0	0.0	0.0	0.0

It is observed that the bounds of the interval solution match the corresponding bounds of combinatorial solution exactly

Table 3 Clamped rectangular plate(4×4)- selected displacements and rotations of the plate for 1% uncertainty of E (Case-B)

Method	$w_{13} \times 10^3 (m)$		$\theta_x \times 10^3$(radians) at node 7		$\theta_y \times 10^3$(radians) at node 12	
	Lower	Upper	Lower	Upper	Lower	Upper
Combinatorial	-1.82260	-1.80446	-1.04167	-1.02805	2.67626	2.70315
Interval	-1.82302	-1.80395	-1.04455	-1.02510	2.67482	2.70446
Error%	0.023	0.028	0.276	0.287	0.054	0.048

It is observed that the bounds of the interval solution sharply enclose the corresponding bounds of combinatorial solution

Table 4 Clamped rectangular plate(4×4)- selected displacements and rotations of the plate for 10% uncertainty of load and 1% uncertainty of E (Case–C)

Method	$w_{13}\times10^3$(m)		$\theta_x \times10^3$(radians) at node 7		$\theta_y \times10^3$(radians) at node 12	
	Lower	Upper	Lower	Upper	Lower	Upper
MCS	-1.87530	-1.74648	-1.07575	-0.97905	2.59554	2.78867
Interval	-1.91180	-1.71030	-1.10937	-0.94955	2.54020	2.84412
Error%	1.946	2.072	3.125	3.013	2.132	1.988

Combinatorial solution is impractical for this case as it requires 2^{32} = 4294967296 combinations. Thus Monte Carlo solution (MCS) is computed. It is observed that the bounds of MCS sharply enclose the bounds of the interval solution from inside

Table 5 Clamped rectangular plate (4×4)- moments at the center of the plate for 10% uncertainty of load (Case-A)				
Method	M_{xx} (kN) at node 13		$M_{yy} \times 10^3$(kN) at node 13	
	Lower	Upper	Lower	Upper
Comb	-2653.612	-2400.887	-1421.684	-1243.397
Interval	-2653.612	-2400.887	-1421.684	-1243.397
Error%	0.0	0.0	0.0	0.0

It is observed that the bounds of the interval solution match the corresponding bounds of combinatorial solution exactly

Table 6 Clamped rectangular plate (4×4)- moments at the center of the plate for 1% uncertainty of E (Case-B)				
Method	M_{xx} (kN) at node 13		$M_{yy} \times 10^3$ (kN) at node 13	
	Lower	Upper	Lower	Upper
Comb	-2546.794	-2507.773	-1343.636	-1321.502
Interval	-2561.199	-2493.300	-1358.769	-1306.311
Error%	0.566	0.577	1.126	1.150

It is observed that the bounds of the interval solution sharply enclose the corresponding bounds of combinatorial solution

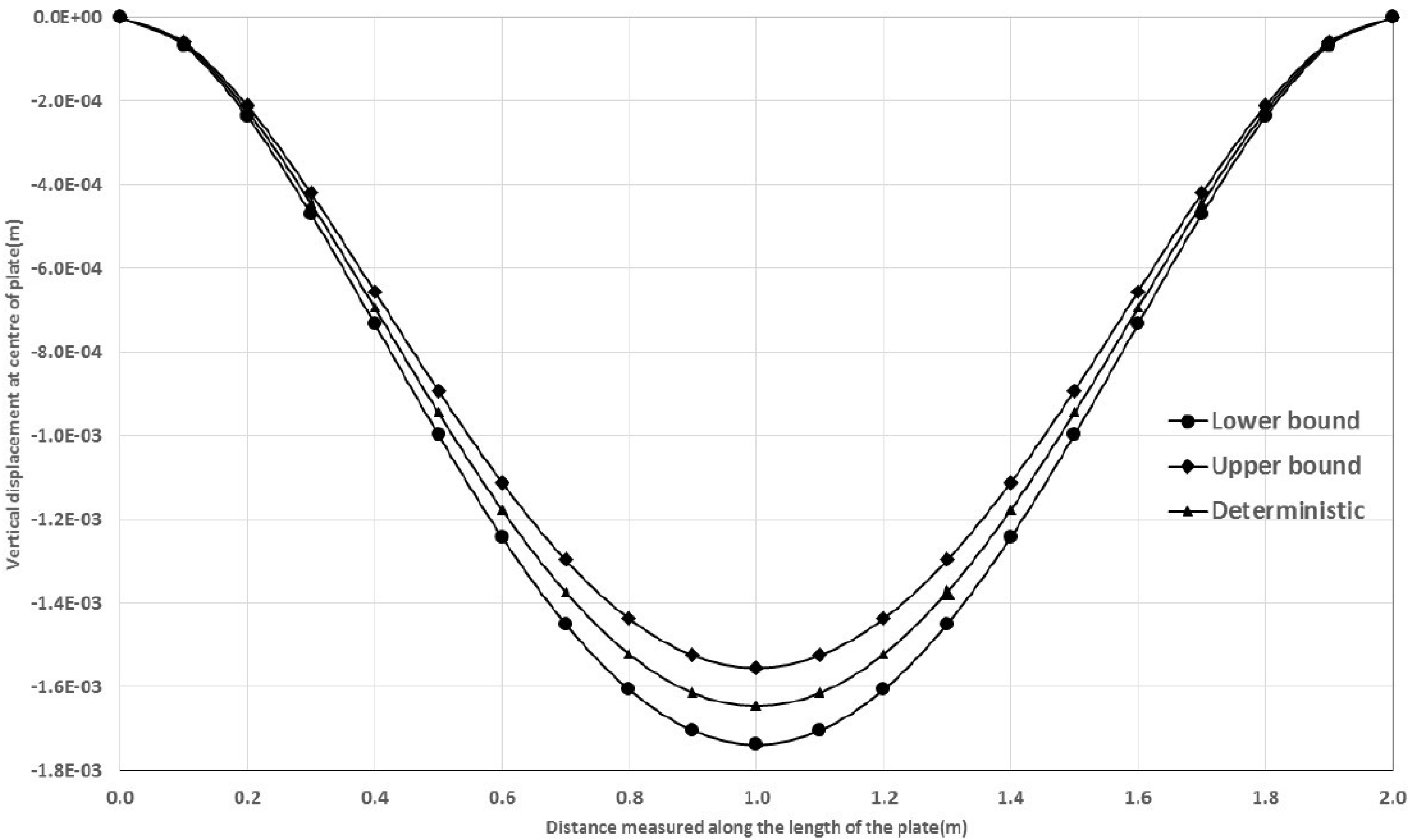
Table 7 Clamped rectangular plate (4x4)- moments at the center of the plate for 10% uncertainty of load and 1% uncertainty of E (Case-C)

Method	M_{xx} (kN) at node 13		$M_{yy} \times 10^3$(kN) at node 13	
	Lower	Upper	Lower	Upper
MCS	-2612.438	-2425.942	-1383.487	-1259.684
Interval	-2678.945	-2358.225	-1434.310	-1211.391
Error%	2.546	2.791	3.674	3.834

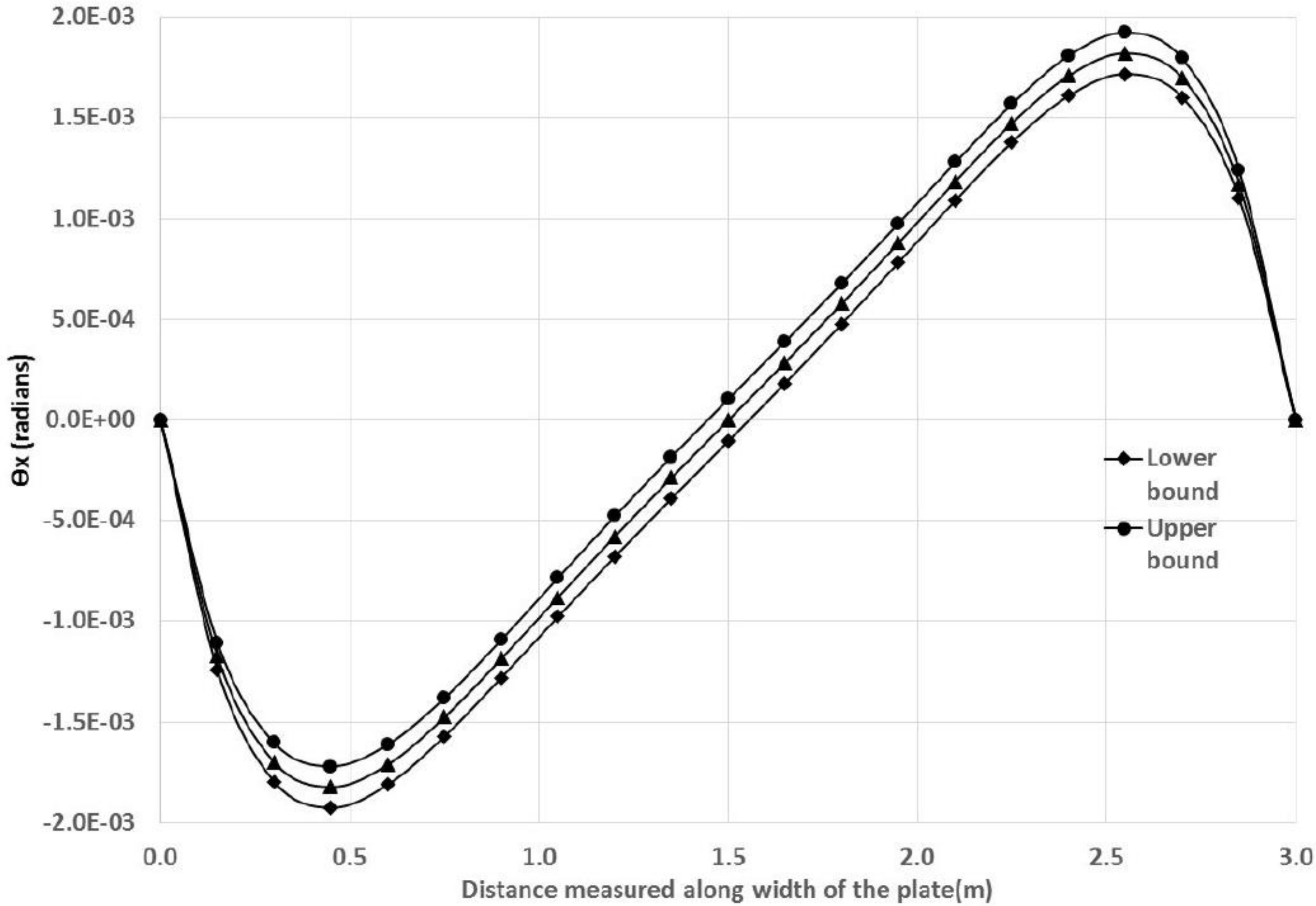
It is observed that the bounds of MCS sharply enclose the bounds of the interval solution from inside

Example problem – Rectangular plate with 20x20 discretization

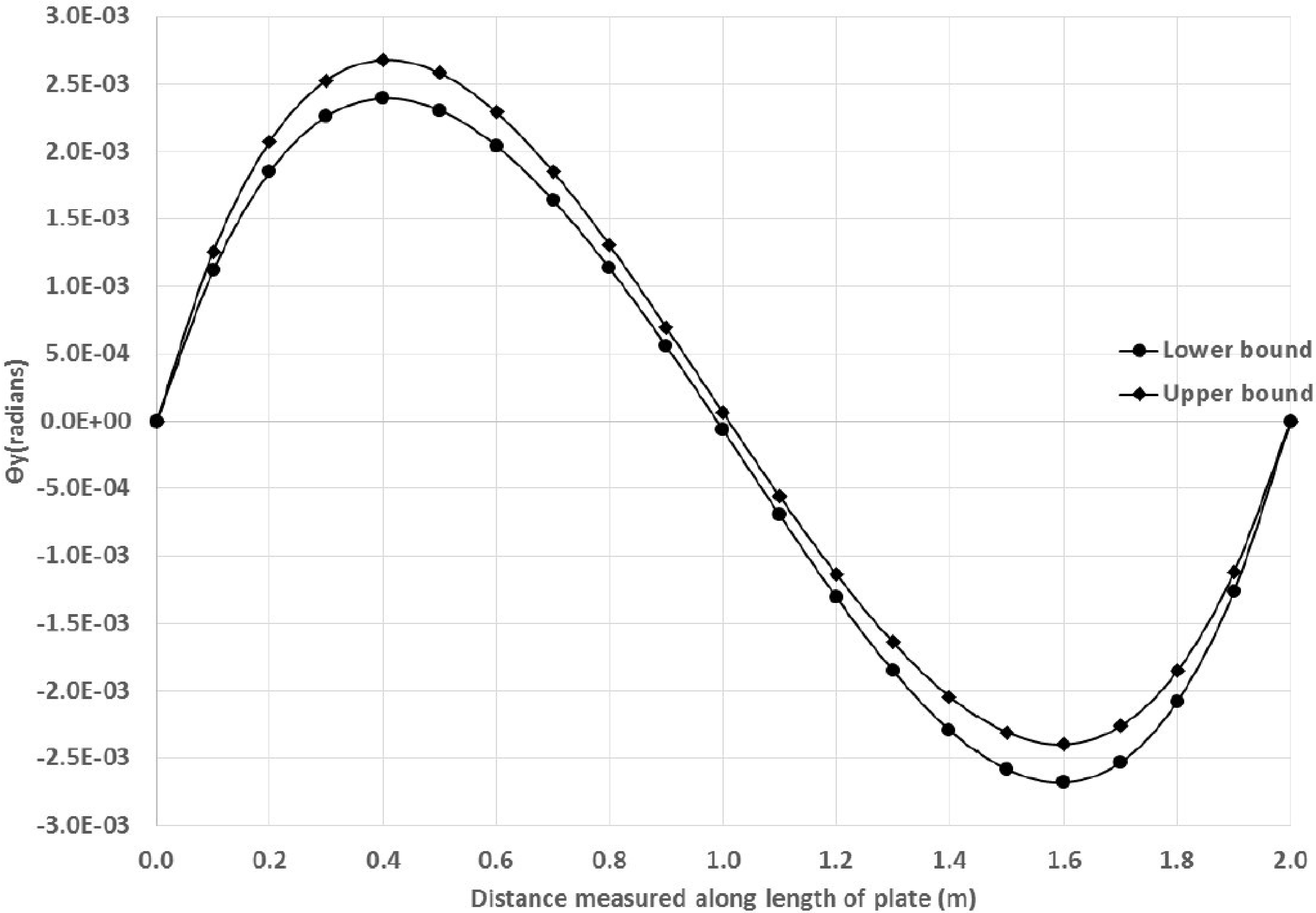
- After validating the results for the example problem with4x4 discretization, results are computed for example problem with 20x20 discretization.
- For all results of displacements, rotations and moments, it is observed that the bounds of MCS sharply enclose the bounds of the interval solution from inside



Variation of vertical displacement along length of the plate with 10% load uncertainty and 1 percent uncertainty in E



Variation of θx along the width of the plate with 10% load uncertainty and 1 percent uncertainty in E



Variation of θy along length of the plate with 10% load uncertainty and 1 percent uncertainty in E

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Table 8 Clamped rectangular plate (20×20)– displacements at the center of the plate for 10% uncertainty of load (Case-A)

Method	$w_{221} \times 10^3$(m)		$\theta_x \times 10^3$(radians) at node 111		$\theta_y \times 10^3$(radians) at node 216	
	Lower	Upper	Lower	Upper	Lower	Upper
MCS	-1.65855	-1.63336	-8.43218	-8.25503	2.42576	2.46399
Interval	-1.72747	-1.56295	-8.85896	-7.81985	2.32204	2.56667
Error%	4.155	4.311	5.061	5.272	4.276	4.167

Table 9 Clamped rectangular plate (20×20)– displacements center of the plate for 1% uncertainty of E (Case-B)

Method	$w_{221} \times 10^3$(m)		$\theta_x \times 10^4$(radians) at node 111		$\theta_y \times 10^3$(radians) at node 216	
	Lower	Upper	Lower	Upper	Lower	Upper
MCS	-1.64694	-1.64384	-8.35812	-8.32283	2.44201	2.44683
Interval	-1.65386	-1.63656	-8.42754	-8.25127	2.43025	2.45845
Error%	0.420	0.443	0.831	0.860	0.482	0.475

Table 10 Clamped rectangular plate (20×20)– displacements at the center of the plate for 10% uncertainty of load and 1% uncertainty of E (Case-C)						
Method	$w_{221} \times 10^3$(m)		$\theta_x \times 10^3$(radians) at node 111		$\theta_y \times 10^3$(radians) at node 216	
	Lower	Upper	Lower	Upper	Lower	Upper
MCS	-1.66052	-1.63068	-8.43013	-8.25141	2.42244	2.46768
Interval	-1.73665	-1.55377	-8.95341	-7.72540	2.30699	2.58172
Error%	4.585	4.716	6.207	6.375	4.766	4.621

Table 11 Clamped rectangular plate (20×20)– moments center of the plate for 10% uncertainty of load (Case-A)				
Method	M_{xx} (kN) at node 221		$M_{yy}\times10^3$(kN) at node 221	
	Lower	Upper	Lower	Upper
MCS	-2096.211	-2057.435	-1154.767	-1124.961
Interval	-2180.916	-1973.210	-1204.756	-1077.381
Error%	4.041	4.094	4.329	4.229

Table 12 Clamped rectangular plate (20×20)– moments at the center of the plate for 1% uncertainty of E (Case-B)				
Method	M_{xx} (kN) at node 221		$M_{yy}\times10^{3}$(kN) at node 221	
	Lower	Upper	Lower	Upper
MCS	-2088.860	-2064.535	-1146.630	-1135.780
Interval	-2127.365	-2026.761	-1175.676	-1106.461
Error%	1.843	1.830	2.533	2.581

Table 13 Clamped rectangular plate (20×20)– moments at the center of the plate for 10% uncertainty of load and 1% uncertainty of E (Case-C)				
Method	M_{xx} (kN) at node 221		$M_{yy}\times10^{3}$(kN) at node 221	
	Lower	Upper	Lower	Upper
MCS	-2108.068	-2051.424	-1161.423	-1121.883
Interval	-2234.305	-1919.821	-1242.057	-1040.080
Error%	5.988	6.415	6.943	7.292

Conclusions

- A linear Interval Finite Element Method (IFEM) for structural analysis of thin plates is presented.
- Uncertainty in the applied load and Young's modulus is represented as interval numbers.
- Results are also computed using combinatorial solution and Monte Carlo simulations as appropriate.

Conclusions

- Example problems illustrate the applicability of the present approach to the problem of predicting the structural behavior of thin plates in the presence of uncertainties.

THANK YOU

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